

HOMOGENEOUS PRINCIPAL BUNDLES AND STABILITY

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ABSTRACT. Let G/P be a rational homogeneous variety, where P is a parabolic subgroup of a simple and simply connected linear algebraic group G defined over an algebraically closed field of characteristic zero. A homogeneous principal bundle over G/P is semistable (respectively, polystable) if and only if it is equivariantly semistable (respectively, equivariantly polystable). A stable homogeneous principal H -bundle (E_H, ρ) is equivariantly stable, but the converse is not true in general. If a homogeneous principal H -bundle (E_H, ρ) is equivariantly stable, but E_H is not stable, then the principal H -bundle E_H admits an action ρ' of G such that the pair (E_H, ρ') is a homogeneous principal H -bundle which is not equivariantly stable.

1. INTRODUCTION

Let G be a simple and simply connected linear algebraic group defined over an algebraically closed field k of characteristic zero. Fix a proper parabolic subgroup P of G . Fix a very ample line bundle ξ on the projective variety G/P . Let H be any reductive linear algebraic group defined over k . For any homomorphism

$$\eta : P \longrightarrow H$$

with the property that the image of η is not contained in any proper parabolic subgroup of H , the associated principal H -bundle $G \times^P H$ over G/P is known to be stable with respect to ξ [AzB, page 576, Theorem 2.6].

A homogeneous principal H -bundle on G/P is a principal H -bundle $E_H \longrightarrow G/P$ together with an action of G

$$\rho : G \times E_H \longrightarrow E_H$$

that lifts the left-translation action of G on G/P . It may be mentioned that all homogeneous principal H -bundles over G/P are given by homomorphisms from P to H . Here we consider those homogeneous principal H -bundles over G/P that arise from homomorphisms for which the image is contained in some proper parabolic subgroup of H . We also consider a weaker notion of stability. A homogeneous principal H -bundle (E_H, ρ) over G/P is called equivariantly stable (respectively, equivariantly semistable) if the usual stability condition (respectively, the semistability condition) holds for those reduction of structure groups of E_H that are preserved by the action ρ of G on E_H . Equivariantly polystable homogeneous principal H -bundles are defined similarly.

We show that a homogeneous principal H -bundle (E_H, ρ) over G/P is equivariantly semistable if and only if the principal H -bundle E_H is semistable (Lemma 4.1). Similarly,

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(E_H, ρ) is equivariantly polystable if and only if the principal H -bundle E_H is polystable (Lemma 4.2).

If E_H is stable, then (E_H, ρ) is equivariantly stable. But the converse is not true. However the following weak converse holds (see Theorem 5.1):

Theorem 1.1. *Let (E_H, ρ) be an equivariantly stable homogeneous principal H -bundle over G/P such that the principal H -bundle E_H is not stable. Then there is an action of G on E_H*

$$\rho' : G \times E_H \longrightarrow E_H$$

such that the following two hold:

- (1) the pair (E_H, ρ') is a homogeneous principal H -bundle, and
- (2) the homogeneous principal H -bundle (E_H, ρ') is not equivariantly stable.

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2. PRELIMINARIES

Let k be an algebraically closed field of characteristic zero. Let G be a simple and simply connected linear algebraic group defined over the field k . Fix a proper parabolic subgroup

$$P \subset G.$$

So the quotient

$$(2.1) \quad M := G/P$$

is an irreducible smooth projective variety defined over k . The quotient map

$$(2.2) \quad f_0 : G \longrightarrow G/P$$

defines a principal P -bundle over M . The left translation action of G on itself defines a homomorphism

$$(2.3) \quad \phi : G \longrightarrow \text{Aut}(M).$$

Fix a very ample line bundle ξ on M . It is known that any ample line bundle on M is very ample. The degree of any torsionfree coherent sheaf on M will be defined using ξ . More precisely, for any torsionfree coherent sheaf F on M , the degree of F is defined to be the degree of the restriction of F to any smooth complete intersection curve on M obtained by intersecting hyperplanes from the complete linear system $|\xi|$. Let F be a vector bundle defined over a nonempty Zariski open dense subset $U \subseteq G/P$ such that the codimension of the complement $(G/P) \setminus U$ is at least two. Then the direct image $\iota_* F$ is a torsionfree coherent sheaf on G/P , where $\iota : U \hookrightarrow G/P$ is the inclusion map. For such a coherent sheaf F define

$$\text{degree}(F) := \text{degree}(\iota_* F).$$

Let H be a connected reductive linear algebraic group defined over the field k . Let Q be a proper parabolic subgroup of H , and let λ be a character of Q which is trivial on the connected component of the center of H containing the identity element. Such a character λ is called *strictly anti-dominant* if the associated line bundle $L_\lambda = H \times^Q k$ over H/Q is ample.

A principal H -bundle E_H over M is called *stable* (respectively, *semistable*) if for every triple of the form (Q, E_Q, λ) , where

- $Q \subsetneq H$ is a proper parabolic subgroup, and

$$(2.4) \quad E_Q \subset E_H$$

is a reduction of structure group of E_H to Q over some nonempty Zariski open subset $U \subset G/P$ such that the codimension of the complement $(G/P) \setminus U$ is at least two, and

- λ is some strictly anti-dominant character of Q (see the above definition of an anti-dominant character),

the inequality

$$\text{degree}(E_Q(\lambda)) > 0$$

(respectively, $\text{degree}(E_Q(\lambda)) \geq 0$) holds, where $E_Q(\lambda)$ is the line bundle over U associated to the principal Q -bundle E_Q for the character λ of Q .

In order to be able to decide whether a given principal H -bundle E_H is stable (respectively, semistable), it suffices to verify the above strict inequality (respectively, inequality) only for the maximal proper parabolic subgroups of H . More precisely, E_H is stable (respectively, semistable) if and only if for every pair of the form (Q, σ) , where

- $Q \subset H$ is a proper maximal parabolic subgroup, and
- σ is a reduction of structure group of E_H to Q

$$(2.5) \quad \sigma : U \longrightarrow E_H/Q$$

over some Zariski open dense subset $U \subset G/P$ such that the codimension of the complement $(G/P) \setminus U$ is at least two,

the inequality

$$(2.6) \quad \text{degree}(\sigma^* T_{\text{rel}}) > 0$$

$$(2.7) \quad (\text{respectively, } \text{degree}(\sigma^* T_{\text{rel}}) \geq 0)$$

holds, where T_{rel} is the relative tangent bundle over E_H/Q for the natural projection $E_H/Q \longrightarrow G/P$. (See [Ra, page 129, Definition 1.1] and [Ra, page 131, Lemma 2.1].)

Let E_H be a principal H -bundle over G/P . A reduction of structure group of E_H

$$E_Q \subset E_H$$

to some parabolic subgroup $Q \subset H$ is called *admissible* if for each character λ of Q trivial on the center of H , the degree of the associated line bundle $E_Q(\lambda) = E_Q \times^Q k$ is zero [Ra, page 307, Definition 3.3].

The unipotent radical of a parabolic subgroup $Q \subset H$ will be denoted by $R_u(Q)$. The quotient group

$$L(Q) := P/R_u(Q),$$

which is called the *Levi quotient* of Q , is a connected reductive linear algebraic group defined over k . A *Levi subgroup* of Q is a closed connected reductive subgroup

$$L' \subset Q$$

such that the composition homomorphism

$$L' \hookrightarrow Q \longrightarrow L(Q)$$

is an isomorphism (here $Q \longrightarrow L(Q)$ is the quotient map). (See [Bo, page 158, § 11.22] and [Hu, page 184, § 30.2].) The notation $L(Q)$ will also be used for denoting a Levi subgroup of Q .

A principal H -bundle E_H over G/P is called *polystable* if either E_H is stable, or there is a proper parabolic subgroup $Q \subset H$ and a reduction of structure group over G/P

$$E_{L(Q)} \subset E_H$$

to a Levi subgroup $L(Q)$ of Q such that the following two conditions hold:

- (1) the principal $L(Q)$ -bundle $E_{L(Q)}$ is stable, and
- (2) the reduction of structure group of E_H to Q obtained by extending the structure group of $E_{L(Q)}$ using the inclusion of $L(Q)$ in Q is admissible.

Let H' be any linear algebraic group defined over k .

Definition 2.1. A *homogeneous* principal H' -bundle over G/P is a principal H' -bundle

$$(2.8) \quad f : E_{H'} \longrightarrow G/P$$

together with an action of G

$$(2.9) \quad \rho : G \times E_{H'} \longrightarrow E_{H'}$$

such that the following two conditions hold:

- (1) $f \circ \rho(g, z) = \phi(g)(f(z))$ for all $(g, z) \in G \times E_{H'}$, where ϕ and f are defined in Eq. (2.3) and Eq. (2.8) respectively, and
- (2) the actions of G and H' on $E_{H'}$ commute.

Let H be a connected reductive linear algebraic group defined over k .

Definition 2.2. A homogeneous principal H -bundle (E_H, ρ) is called *equivariantly stable* (respectively, *equivariantly semistable*) if the condition in the definition of stability (respectively, semistability) holds for all E_Q as in Eq. (2.4) that are left invariant by the action ρ of G on E_H .

Similarly, a homogeneous principal H -bundle (E_H, ρ) is called *equivariantly polystable* if either E_H is equivariantly stable, or there is a proper parabolic subgroup $Q \subset H$ and a reduction of structure group over G/P

$$E_{L(Q)} \subset E_H$$

to a Levi subgroup $L(Q)$ of Q such that the following three conditions hold:

- (1) the action of G on E_H leaves $E_{L(Q)}$ invariant,
- (2) the principal $L(Q)$ -bundle $E_{L(Q)}$ is equivariantly stable, and
- (3) the reduction of structure group of E_H to Q obtained by extending the structure group of $E_{L(Q)}$ using the inclusion of $L(Q)$ in Q is admissible.

Remark 2.3. A homogeneous principal H -bundle (E_H, ρ) is equivariantly stable if the inequality in Eq. (2.6) holds for all σ as in Eq. (2.5) that are invariant under the action of G on E_H/Q defined by ρ .

Similarly, a homogeneous principal H -bundle (E_H, ρ) is equivariantly semistable if the inequality in Eq. (2.7) holds for all σ as in Eq. (2.5) that are invariant under the action of G on E_H/Q defined by ρ .

3. A CRITERION FOR HOMOGENEOUS PRINCIPAL BUNDLES

If $(E_{H'}, \rho)$ is a homogeneous principal H' -bundle over G/P , then for each point $g \in G$, the pulled back principal H' -bundle $\phi(g)^*E_{H'}$ is isomorphic to $E_{H'}$, where ϕ is the homomorphism in Eq. (2.3). Indeed, the automorphism of the variety $E_{H'}$ defined by $z \mapsto \rho(g, z)$ gives an isomorphism of the principal H' -bundles $E_{H'} \rightarrow \phi(g)^*E_{H'}$.

The following proposition asserts a converse of the above observation.

Proposition 3.1. *Let H' be a linear algebraic group defined over k . Let*

$$\gamma : E_{H'} \longrightarrow G/P$$

*be a principal H' -bundle such that for each point $g \in G$, the pulled back principal G -bundle $\phi(g)^*E_{H'}$ is isomorphic to $E_{H'}$, where ϕ is the homomorphism in Eq. (2.3). Then there is an action of G on $E_{H'}$*

$$\rho : G \times E_{H'} \longrightarrow E_{H'}$$

such that the pair $(E_{H'}, \rho)$ is a homogeneous principal H' -bundle.

Proof. Let \mathcal{A} denote the group of automorphisms of the principal H' -bundle $E_{H'}$. So \mathcal{A} consists of all automorphisms of the variety $E_{H'}$

$$h : E_{H'} \longrightarrow E_{H'}$$

such that

- $\gamma \circ h = \gamma$, and
- h commutes with the action of H' on $E_{H'}$.

We will show that \mathcal{A} is a linear algebraic group defined over k .

Fix a finite dimensional faithful representation

$$(3.1) \quad \tau : H' \longrightarrow \mathrm{GL}(V)$$

of H' . Let $E_V := E_{H'} \times^{H'} V$ be the vector bundle over G/P associated to $E_{H'}$ for this H' -module V . Let $\mathrm{Aut}(E_V)$ denote the group of all automorphisms of the vector bundle E_V . We note that

$$\mathrm{Aut}(E_V) \hookrightarrow H^0(G/P, E_V \otimes E_V^*) .$$

Using this inclusion, $\mathrm{Aut}(E_V)$ has the structure of a linear algebraic group defined over k . Any automorphism of the principal H' -bundle E'_H yields an automorphism of the associated vector bundle E_V . Since τ in Eq. (3.1) is injective, it follows that \mathcal{A} is a closed subgroup of $\mathrm{Aut}(E_V)$. Hence \mathcal{A} is a linear algebraic group defined over k .

Let $\tilde{\mathcal{A}}$ denote the group of all pairs of the form (g, h) , where $g \in G$, and

$$h : E_{H'} \longrightarrow E_{H'}$$

is an automorphism of the variety $E_{H'}$ satisfying the following two conditions:

- (1) $\gamma \circ h = \phi(g) \circ \gamma$, and
- (2) h commutes with the action of H' on $E_{H'}$.

The group operation on $\tilde{\mathcal{A}}$ is:

$$(g_1, h_1)(g_2, h_2) := (g_1 \circ g_2, h_1 \circ h_2) .$$

We will show that $\tilde{\mathcal{A}}$ is also a linear algebraic group defined over k .

Let

$$(3.2) \quad \phi_0 : G \times M \longrightarrow M := G/P$$

be the left action defined by ϕ in Eq. (2.3). Let

$$(3.3) \quad p_2 : G \times M \longrightarrow M$$

be the projection to the second factor. Let \mathcal{S} denote the sheaf of isomorphisms from the principal H' -bundle $p_2^* E_{H'}$ to the principal H' -bundle $\phi_0^* E_{H'}$ over $G \times M$, where ϕ_0 and p_2 are defined in Eq. (3.2) and Eq. (3.3) respectively. Now consider the direct image

$$\tilde{\mathcal{S}} := p_{1*} \mathcal{S}$$

over G , where p_1 as before is the projection of $G \times M$ to G . Comparing the definitions $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{A}}$ it follows immediately that $\tilde{\mathcal{S}}$ is identified with $\tilde{\mathcal{A}}$.

As before, let E_V denote the vector bundle associated to $E_{H'}$ for the faithful H' -module V in Eq. (3.1). The total space of $\tilde{\mathcal{S}}$ is naturally embedded in the total space of the vector bundle

$$(3.4) \quad p_{1*}((\phi_0^* E_V) \otimes p_2^* E_V^*) \longrightarrow G ,$$

where ϕ_0 and p_2 are defined in Eq. (3.2) and Eq. (3.3) respectively, and p_1 is the projection of $G \times M$ to G . Using this embedding, the total space of $\tilde{\mathcal{S}}$ gets a structure of a scheme

defined over k . Consequently, the identification of $\tilde{\mathcal{A}}$ with $\tilde{\mathcal{S}}$ makes $\tilde{\mathcal{A}}$ a scheme defined over k . The group operations (multiplication and inverse maps) are algebraic. Hence $\tilde{\mathcal{A}}$ is an algebraic group defined over k .

Since G is an affine variety, the total space of the vector bundle $p_{1*}((\phi_0^*E_V) \otimes p_2^*E_V^*)$ in Eq. (3.4) is also an affine variety. So $\tilde{\mathcal{A}}$ is an affine scheme. Therefore, we conclude that $\tilde{\mathcal{A}}$ is a linear algebraic group defined over k .

Let

$$(3.5) \quad p : \tilde{\mathcal{A}} \longrightarrow G$$

be the homomorphism defined by $(g, h) \mapsto g$. Let

$$(3.6) \quad I : \mathcal{A} \longrightarrow \tilde{\mathcal{A}}$$

be the homomorphism defined by $h \mapsto (e, h)$, where $e \in G$ is the identity element. Since $\phi(g)^*E_{H'}$ is isomorphic to $E_{H'}$ for all $g \in G$, the homomorphism p in Eq. (3.5) is surjective. Hence we have a short exact sequence of groups

$$(3.7) \quad e \longrightarrow \mathcal{A} \xrightarrow{I} \tilde{\mathcal{A}} \xrightarrow{p} G \longrightarrow e,$$

where I is defined in Eq. (3.6).

We will show that the short exact sequence in Eq. (3.7) is right split, or in other words, there is a homomorphism

$$(3.8) \quad \psi : G \longrightarrow \tilde{\mathcal{A}}$$

such that $p \circ \psi = \text{Id}_G$.

To prove this, let $\tilde{\mathcal{A}}_0$ denote the connected component of $\tilde{\mathcal{A}}$ containing the identity element. Let

$$(3.9) \quad \mathcal{G} \subset \tilde{\mathcal{A}}_0$$

be a maximal connected reductive subgroup of $\tilde{\mathcal{A}}_0$ [Mo, page 217, Theorem 7.1]. (As $\tilde{\mathcal{A}}_0$ is connected, from [Mo, page 217, Theorem 7.1] we know that any two maximal connected reductive subgroups of it are conjugate.) Therefore, the commutator subgroup

$$\mathcal{G}' := [\mathcal{G}, \mathcal{G}] \subset \mathcal{G}$$

is semisimple, where \mathcal{G} is constructed in Eq. (3.9). The homomorphism p in Eq. (3.5) is surjective, and G is simple. Hence the restriction

$$p' := p|_{\mathcal{G}'} : \mathcal{G}' \longrightarrow G$$

is also surjective. Express the semisimple group \mathcal{G}' as a quotient of a product of simple and simply connected groups by a finite group. So

$$\mathcal{G}' = (\prod_{i=1}^n G_i)/\Gamma,$$

where each G_i is a simple and simply connected linear algebraic group defined over k , and Γ is a finite group contained in the center of $\prod_{i=1}^n G_i$. Let

$$q' : \prod_{i=1}^n G_i \longrightarrow \mathcal{G}'$$

be the quotient map. Since G is simple and simply connected, and p' is surjective, there is some $i_0 \in [1, n]$ such that the homomorphism

$$p_0 := (p' \circ q')|_{G_{i_0}} : G_{i_0} \longrightarrow G$$

is an isomorphism.

The homomorphism

$$\psi := q' \circ p_0^{-1} : G \longrightarrow \mathcal{G}' \hookrightarrow \tilde{\mathcal{A}}$$

clearly satisfies the splitting condition

$$p \circ \psi = \text{Id}_G.$$

Fix a homomorphism ψ as in Eq. (3.8) such that $p \circ \psi = \text{Id}_G$.

Now we have an action of G on $E_{H'}$

$$\rho : G \times E_{H'} \longrightarrow E_{H'}$$

defined by

$$(g, z) \longmapsto \psi(g)(z) \in (E_{H'})_{\phi(g)(\gamma(z))},$$

where ϕ and ψ are the maps in Eq. (2.3) and Eq. (3.8) respectively (the map γ is as in the statement of the proposition). It is straight-forward to check that ρ satisfies the two conditions in Definition 2.1. This completes the proof of the proposition. \square

4. SEMISTABLE AND POLYSTABLE HOMOGENEOUS PRINCIPAL BUNDLES

Let H be a connected reductive linear algebraic group defined over k . Let (E_H, ρ) be a homogeneous principal H -bundle over G/P .

Lemma 4.1. *The principal H -bundle E_H is semistable if and only if (E_H, ρ) is equivariantly semistable.*

Proof. If E_H is semistable, then clearly (E_H, ρ) is equivariantly semistable. To prove the converse, assume that E_H is not semistable. Then E_H admits a unique Harder–Narasimhan reduction

$$E_Q \subset E_H$$

that contradicts the semistability condition of E_H (see [BH, page 211, Theorem 4.1]). From the uniqueness of E_Q it follows immediately that the action of G on E_H leaves E_Q invariant. Therefore, E_H is not equivariantly semistable. This completes the proof of the lemma. \square

Lemma 4.2. *Let (E_H, ρ) be a homogeneous principal H -bundle over G/P . The principal H -bundle E_H is polystable if and only if (E_H, ρ) is equivariantly polystable.*

Proof. First assume that E_H is polystable. We will show that (E_H, ρ) is equivariantly polystable. Since the characteristic of the field k is zero, it suffices to prove this under the assumption that $k = \mathbb{C}$. We assume that $k = \mathbb{C}$.

Fix a maximal compact subgroup

$$(4.1) \quad K \subset G.$$

Fix a Kähler form ω on G/P satisfying the following two conditions:

- the action of K on G/P (given by ϕ in Eq. (2.3)) preserves ω , and
- the cohomology class in $H^2(G/P, \mathbb{C})$ represented by the closed form ω coincides with $c_1(\xi)$, where ξ is the fixed ample line bundle on G/P .

Since the principal H -bundle E_H is polystable, it admits a unique Einstein–Hermitian connection with respect to ω [RS, page 24, Theorem 1], [AnB, page 221, Theorem 3.7]. Although the uniqueness of an Einstein–Hermitian connection is well known, we will explain it here because neither of [RS] and [AnB] explicitly mentions it.

On a vector bundle W admitting an Einstein–Hermitian connection, there is exactly one Einstein–Hermitian connection. Indeed, if W is indecomposable, then this is proved in [Do, page 12, Corollary 9 (i)]; the general case, where W is a direct sum of indecomposable vector bundles, follows from this and [Si, page 878, Proposition 3.3]. Let

$$Z_0(H) \subset H$$

be the connected component, containing the identity element, of the center of H . Take any homomorphism

$$(4.2) \quad \beta : H \longrightarrow \mathrm{GL}(n, \mathbb{C})$$

that takes $Z_0(H)$ to the center of $\mathrm{GL}(n, \mathbb{C})$. Let

$$E_H(\beta) := E_H \times^H \mathbb{C}^n \longrightarrow G/P$$

be the vector bundle associated to the principal H -bundle E_H for β and the standard representation of $\mathrm{GL}(n, \mathbb{C})$. The condition on β ensures that the connection $\nabla(E_H(\beta))$ on $E_H(\beta)$ induced by an Einstein–Hermitian connection $\nabla(E_H)$ on E_H is also Einstein–Hermitian. Hence $\nabla(E_H(\beta))$ is the unique Einstein–Hermitian connection on the vector bundle $E_H(\beta)$.

Fix characters

$$\chi_i : H \longrightarrow \mathbb{C}^*,$$

$i \in [1, n]$, such that the map

$$\prod_{i=1}^n \chi_i : H/[H, H] \longrightarrow (\mathbb{C}^*)^n$$

is an embedding. Let \mathfrak{h} be the Lie algebra of H . Let ∇_1 and ∇_2 be two connections on the principal H -bundle E_H satisfying the following conditions:

- the connections on the adjoint vector bundle $\mathrm{ad}(E_H) := E_H \times^H \mathfrak{h}$ induced by ∇_1 and ∇_2 coincide, and

- for each $i \in [1, n]$, the connections on the associated line bundle $E_H \times^{\chi_i} \mathbb{C}$ induced by ∇_1 and ∇_2 coincide.

Then it is straight forward to check that ∇_1 coincides with ∇_2 .

Now setting the above representations for β in Eq. (4.2) we conclude that E_H admits at most one Einstein–Hermitian connection.

It should be clarified that although the Einstein–Hermitian connection is unique, the Einstein–Hermitian metric (which is a C^∞ reduction of structure group of the principal H –bundle to a maximal compact subgroup of H) is not unique. Any two Einstein–Hermitian reductions on a given principal H –bundle differ by the translation by an element of $Z_0(H)/K(Z_0(H))$, where $K(Z_0(H)) \subset Z_0(H)$ is the maximal compact subgroup.

Let $\nabla(E_H)$ denote the unique Einstein–Hermitian connection on E_H . From the uniqueness of $\nabla(E_H)$ it follows immediately that the action of the group K in Eq. (4.1) on E_H (given by ϕ in Eq. (2.3)) preserves the connection $\nabla(E_H)$.

In [RS] it is proved that a principal bundle admitting an Einstein–Hermitian connection is polystable (see [RS, § 4, pages 28–29]). Using the fact that the action of K on E_H preserves the Einstein–Hermitian connection $\nabla(E_H)$, this proof in [RS] gives that E_H is equivariantly polystable for the action of K on E_H . Since K is a maximal compact subgroup of G , any K –invariant holomorphic reduction of structure group of E_H is automatically G –invariant. Therefore, we now conclude that the homogeneous principal H –bundle (E_H, ρ) is equivariantly polystable for the action of G on E_H .

To prove the converse, assume that (E_H, ρ) is equivariantly polystable. Let

$$E_{L(Q)} \subset E_H$$

be a G –invariant minimal Levi reduction of the structure group [BP, page 56, Theorem 1.3]. The action of G on $E_{L(Q)}$ induced by ρ will also be denoted by ρ . We note that the homogeneous principal $L(Q)$ –bundle $(E_{L(Q)}, \rho)$ is equivariantly stable because $E_{L(Q)}$ is a G –invariant minimal Levi reduction of E_H .

Since $(E_{L(Q)}, \rho)$ is equivariantly stable, from Lemma 4.1 it follows that the principal $L(Q)$ –bundle $E_{L(Q)}$ is semistable. Let $\text{ad}(E_{L(Q)})$ be the adjoint vector bundle of $E_{L(Q)}$. We recall that $\text{ad}(E_{L(Q)})$ is the vector bundle over G/P associated to the principal $L(Q)$ –bundle $E_{L(Q)}$ for the adjoint action of $E_{L(Q)}$ on its own Lie algebra. The adjoint vector bundle $\text{ad}(E_{L(Q)})$ is semistable because the principal $L(Q)$ –bundle $E_{L(Q)}$ is semistable [RR, page 285, Theorem 3.18]. Let

$$(4.3) \quad W_0 \subset \text{ad}(E_{L(Q)})$$

be the socle of the semistable vector bundle (see [HL, page 23, Lemma 1.5.5]). Since the vector bundle $\text{ad}(E_{L(Q)})$ is homogeneous, it follows that W_0 is actually a subbundle of $\text{ad}(E_{L(Q)})$.

We will show that the principal $L(Q)$ –bundle $E_{L(Q)}$ is polystable.

To prove this by contradiction, assume that $E_{L(Q)}$ is not polystable. Therefore, the semistable vector bundle $\text{ad}(E_{L(Q)})$ is not polystable. Consequently, the subbundle W_0 in Eq. (4.3) is a proper one.

In [AnB], using W_0 a unique reduction of structure group of E_H

$$(4.4) \quad E_{Q_0} \subset E_{L(Q)}$$

to a certain proper parabolic subgroup

$$(4.5) \quad Q_0 \subsetneq L(Q)$$

is constructed; the parabolic subgroup Q_0 is also constructed using W_0 (see [AnB, page 218, Proposition 2.12]). This reduction E_{Q_0} constructed in [AnB, page 218, Proposition 2.12] is admissible (admissible reductions were defined in Section 2). From the uniqueness of E_{Q_0} in Eq. (4.4) it follows immediately that the action ρ of G on $E_{L(Q)}$ leaves the subvariety E_{Q_0} invariant. Since E_{Q_0} is an admissible reduction of structure group of $E_{L(Q)}$ which is left invariant by the action of G on $E_{L(Q)}$, and $E_{L(Q)}$ is equivariantly stable, it follows that $Q_0 = L(Q)$. But this contradicts Eq. (4.5).

Consequently, $W_0 = \text{ad}(E_{L(Q)})$. Therefore, we conclude that the principal $L(Q)$ -bundle $E_{L(Q)}$ is polystable.

Since the principal $L(Q)$ -bundle $E_{L(Q)}$ is polystable, it follows that the principal H -bundle E_H is polystable. This completes the proof of the lemma. \square

We note that the analog of Lemma 4.1 and Lemma 4.2 for stable principal bundles is not valid. In other words, there are equivariantly stable homogeneous principal H -bundles (E_H, ρ) such that E_H is not stable.

To construct such an example, take a pair (H, η) , where H is a connected reductive nonabelian linear algebraic group defined over k , and

$$(4.6) \quad \eta : G \longrightarrow H$$

is a homomorphism satisfying the following condition: the image $\eta(G)$ is not contained in any proper parabolic subgroup of H . For example, we may take $H = G$ and $\eta = \text{Id}_G$.

Let E_H be the trivial principal H -bundle $M \times H$ over $M = G/P$. Since H is not abelian, and E_H is trivial, it follows that the principal H -bundle E_H is not stable.

We will construct an action of G on E_H .

Consider the action of G on G/P defined by the homomorphism ϕ in Eq. (2.3) together with the left translation action of G on H using the homomorphism η in Eq. (4.6). Now let ρ denote the diagonal action of G on $E_H = (G/P) \times H$. We will show that the resulting homogeneous principal H -bundle (E_H, ρ) is equivariantly stable.

To prove by contradiction, assume that (E_H, ρ) is not equivariantly stable. Since E_H is trivial, it is polystable. Hence from Lemma 4.2 we know that (E_H, ρ) is equivariantly polystable. Therefore, there is a proper parabolic subgroup $Q \subset H$ and a G -invariant

reduction of structure group

$$(4.7) \quad E_Q \subset E_H$$

over G/P such that

$$(4.8) \quad \text{degree}(\text{ad}(E_H)/\text{ad}(E_Q)) = 0,$$

where $\text{ad}(E_H)$ and $\text{ad}(E_Q)$ are the adjoint vector bundles of E_H and E_Q respectively.

Let q_0 denote the dimension of the group Q .

Since $E_H = M \times H$, the adjoint vector bundle $\text{ad}(E_H)$ is identified with the trivial vector bundle $M \times \mathfrak{h}$, where \mathfrak{h} is the Lie algebra of H . Therefore, the subbundle

$$(4.9) \quad \text{ad}(E_Q) \subset \text{ad}(E_H) = M \times \mathfrak{h}$$

defines a morphism

$$(4.10) \quad \theta : G/P \longrightarrow \text{Gr}(q_0, \mathfrak{h}),$$

where $\text{Gr}(q_0, \mathfrak{h})$ is the Grassmann variety that parametrizes all subspaces of \mathfrak{h} of dimension $q_0 := \dim Q$. The morphism θ in Eq. (4.10) sends any $x \in G/P$ to the subspace $\text{ad}(E_Q)_x \subset \text{ad}(E_H)_x = \mathfrak{h}$. Therefore,

$$\text{ad}(E_H)/\text{ad}(E_Q) = \theta^*\mathcal{Q},$$

where $\mathcal{Q} \longrightarrow \text{Gr}(q_0, \mathfrak{h})$ is the tautological quotient bundle (the fiber of \mathcal{Q} over any point of $\text{Gr}(q_0, \mathfrak{h})$ is the quotient of \mathfrak{h} by the corresponding subspace). Hence from Eq. (4.8),

$$(4.11) \quad \text{degree}(\theta^*\mathcal{Q}) = \text{degree}(\theta^*\det(\mathcal{Q})) = 0,$$

where $\det(\mathcal{Q}) = \bigwedge^{\text{top}} \mathcal{Q}$ is the top exterior product of \mathcal{Q} . The line bundle $\det(\mathcal{Q})$ over $\text{Gr}(q_0, \mathfrak{h})$ is ample. Hence from Eq. (4.11) it follows that θ is a constant map. Therefore, there is a subspace

$$(4.12) \quad V_0 \subset \mathfrak{h}$$

such that the subbundle $\text{ad}(E_Q)$ in Eq. (4.9) coincides with $M \times V_0 \subset M \times \mathfrak{h}$.

Let \mathfrak{q} be the Lie algebra of Q . Since $\text{ad}(E_Q)$ is the adjoint vector bundle of E_Q it follows that the subspace V_0 in Eq. (4.12) is a conjugate of the subspace $\mathfrak{q} \subset \mathfrak{h}$. Therefore, V_0 is the Lie algebra of a parabolic subgroup

$$(4.13) \quad Q_0 \subset H$$

which is a conjugate of Q .

Recall the given condition that the action of G on E_H leaves E_Q invariant. This implies that the adjoint action of G on \mathfrak{h} , defined using the homomorphism η in Eq. (4.6), leaves the subspace V_0 invariant. Since Q_0 is a parabolic subgroup (see Eq. (4.13)), and V_0 is the Lie algebra of Q_0 , the normalizer of V_0 inside H coincides with Q_0 (see [Bo, page 154, Theorem 11.16], [Hu, page 179, Theorem (c)]). Consequently, we have

$$\eta(G) \subset Q_0.$$

But this contradicts the given condition that $\eta(G)$ is not contained in any proper parabolic subgroup of H . Thus there is no G -invariant reduction as in Eq. (4.7).

Therefore, we conclude that the homogeneous principal H -bundle (E_H, ρ) is equivariantly stable.

We note that if we set $H = \mathrm{GL}(n, k)$, then the above example gives a counter-example to Corollary 2.11 in [Ro].

5. STABLE HOMOGENEOUS PRINCIPAL BUNDLES

A homogeneous principal H -bundle (E_H, ρ) over G/P is clearly equivariantly stable if E_H is stable. The following theorem is a converse of this.

Theorem 5.1. *Let (E_H, ρ) be an equivariantly stable homogeneous principal H -bundle over G/P , where H is a connected reductive linear algebraic group defined over k , such that the principal H -bundle E_H is not stable. Then there is an action*

$$\rho' : G \times E_H \longrightarrow E_H$$

of G on E_H such that the following two hold:

- (1) *the pair (E_H, ρ') is a homogeneous principal H -bundle, and*
- (2) *the homogeneous principal H -bundle (E_H, ρ') is not equivariantly stable.*

Proof. Since (E_H, ρ) is equivariantly stable, from Lemma 4.2 we know that the principal H -bundle E_H is polystable. Therefore, from the given condition that E_H is not stable it follows immediately that E_H admits a reduction of structure group

$$(5.1) \quad E_{L(Q')} \subset E_H$$

to a Levi subgroup $L(Q')$ of some proper parabolic subgroup Q' of H .

Therefore, there is a natural reduction of structure group of E_H to a Levi subgroup $L(Q)$ of a parabolic subgroup $Q \subset H$

$$(5.2) \quad E_{L(Q)} \subset E_H$$

which has the following property: the subgroup $L(Q) \subset H$ is smallest among all the Levi subgroups of parabolic subgroups of H to which E_H admits a reduction of structure group (see [BBN1, page 230, Theorem 3.2] and [BBN1, page 232, Theorem 3.4]). An alternative construction of this reduction of structure group in Eq. (5.2) is given in [BBN2].

Since $L(Q')$ in Eq. (5.1) is a Levi subgroup of a proper parabolic subgroup of H , the Levi subgroup $L(Q)$ in Eq. (5.2) must be a proper subgroup of H .

It should be mentioned that unlike the two reductions in Lemma 4.1 and Eq. (4.4), the reduction in Eq. (5.2) is not unique. However, the isomorphism class of the principal $L(Q)$ -bundle $E_{L(Q)}$ in Eq. (5.2) is uniquely determined (see [BBN1, page 232, Proposition 3.3]). From this it can be deduced that for each point $g \in G$, the pulled back principal

$L(Q)$ -bundle $\phi(g)^*E_{L(Q)}$ is isomorphic to $E_{L(Q)}$, where ϕ is the homomorphism in Eq. (2.3). Indeed, the pulled back reduction of structure group

$$\phi(g)^*E_{L(Q)} \subset \phi(g)^*E_H$$

is of the type constructed in [BBN1]. The principal H -bundle $\phi(g)^*E_H$ is isomorphic to E_H because (E_H, ρ) is homogeneous. Hence from [BBN1, page 232, Proposition 3.3] it follows immediately that $\phi(g)^*E_{L(Q)}$ is isomorphic to the principal $L(Q)$ -bundle $E_{L(Q)}$.

Now from Proposition 3.1 we know that there is an action of G on $E_{L(Q)}$

$$(5.3) \quad \rho'': G \times E_{L(Q)} \longrightarrow E_{L(Q)}$$

such that the pair $(E_{L(Q)}, \rho'')$ is a homogeneous principal $L(Q)$ -bundle.

Since $E_{L(Q)}$ is a reduction of structure group of E_H , the action ρ'' (see Eq. (5.3)) induces an action of G on E_H . To explain this, first note that E_H is a quotient of $E_{L(Q)} \times H$. Two points (z_1, h_1) and (z_2, h_2) of $E_{L(Q)} \times H$ are identified in the quotient space E_H if and only if there is an element $g \in L(Q)$ such that $(z_2, h_2) = (z_1g, g^{-1}h_1)$. The action ρ'' of G on $E_{L(Q)}$ and the trivial action of G on H together define an action of G on $E_{L(Q)} \times H$. Using the fact that the actions, on $E_{L(Q)}$, of $L(Q)$ and G (defined by ρ'') commute we conclude that the above action of G on $E_{L(Q)} \times H$ descends to an action of G on the quotient space E_H . Let

$$\rho' : G \times E_H \longrightarrow E_H$$

be this descended action. It is now easy to see that this pair (E_H, ρ') is a homogeneous principal H -bundle.

The action ρ' of G on E_H preserves the subvariety

$$E_{L(Q)} \subset E_H$$

in Eq. (5.2). In fact, the restriction of ρ' to $E_{L(Q)}$ coincides with ρ'' . We noted earlier that $L(Q)$ is a proper subgroup of H . Hence the existence of the ρ' invariant reduction $E_{L(Q)} \subset E_H$ proves that (E_H, ρ') is not equivariantly stable. This completes the proof of the theorem. \square

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